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Generating functions for the coupling–recoupling coefficients of SU(2)

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Abstract. A general formula for the generating function of an arbitrary coupling–recoupling coefficient of SU(2) is derived via the Bargmann formalism and graphical calculations. The result is expressed in terms of sums over two finite sets only of subgraphs of the Jucys graph of the coefficient. A new explicit expression for the coefficient is obtained.

Index of notations

We give the main notations and the paragraphs in which they are defined.

§ 1:	CRC: coupling–recoupling coefficient
§ 2:	G: Jucys graph
	k_i, κ_i, λ_i: l indices; τ_i, ξ_i, η_i: corresponding variables
	$L = (l_1, \ldots, l_{n_G})$: array of the n_G l indices of a CRC
	E_G: set of all L
	$G_L = G_{jm\ldots}$: value of the CRC
	N_L: normalization constant equation (1)
	Φ_G: generating function
§ 4:	(ab): branch
	$[a \ldots z]$, $]a \ldots z]$ etc: paths
	$(a \ldots z)$: circuit
	$\mathscr{B}(D)$: set of the elements of diagram D
	K_G, Ω_G: sets of closed and open diagrams of G
	$\mathscr{D}, \mathscr{D}_V, \bar{\mathscr{D}}_V, \mathscr{R}_V, F_G, F'_G, H_G, H'_G, Q_G$: sets of diagrams
§ 5:	$M(D)$, $\epsilon(D)$, $L(D)$, $l_i(D)$: functions of diagram D
§ 7:	$\alpha, \beta, \bar{\beta}$: sets of branches
	a, b, c, d, f: matrices
	v, w: vectors $\lvert d\rvert$: number
§ 8.1:	$\pi(C)$: function of circuit C
§ 8.2:	$S_{t,k}, S'_{t,k}, R_{t,k}, F''_G, U_{t,k}$: sets of diagrams
	ω_t, ω'_t: sets of subscripts.

1. Introduction

A coupling–recoupling coefficient (CRC) of SU(2) is most easily described by its Jucys graph (Jucys and Bandzaitis 1965, El Baz 1969, Bordarier 1970). Generating functions for the $3j$, $6j$ and $9j$ coefficients were obtained by Schwinger (1952, see also Biedenharn and Van Dam 1965 pp 229–79) in a creation and annihilation operator approach. In a treatment of the SU(2) group based on entire function spaces Bargmann (1962, see also Biedenharn and Van Dam 1965, pp 300–16) obtained generating functions for the $3j$ and $6j$ coefficients. Generating functions for some other CRC were derived in the Bargmann scheme: by Wu (1972) for the $9j$ and by Huang and Wu (1974) for the $12j$ and $15j$ coefficients.

In this paper a general formula for the generating function of an arbitrary CRC is derived. For a Jucys graph G we define the generating function of the CRC it represents (§ 2). The significance of graph G is made clear in § 3. We introduce certain graphs drawn on G that we call diagrams of G (§ 4). A monomial $M(D)$ is defined (§ 5) for every diagram D of graph G. The final formula for the generating function is expressed in terms of the sums of $M(D)$ over two finite sets of diagrams of G.

At first, we use the method Bargmann (1962, see also Biedenharn and Van Dam 1965, pp 300–16) used to obtain the generating function of the $6j$ coefficient. A general CRC is obtained essentially from $3j$ coefficients by summations on projection quantum numbers. Its generating function is expressed in terms of integrals of a product of generating functions of $3j$ coefficients. An algebraic expression for the generating function is then obtained (§ 7) by carrying out the integrations. At this point there is a slight difference from Bargmann's method: here all the integrations are carried out at one stroke whereas in Bargmann (1962, see also Biedenharn and Van Dam 1965, pp 300–16), Wu (1972) and Huang and Wu (1974) the integrations are carried out in several steps.

In § 8 the algebraic expression is transcribed in terms of sums over generally infinite sets of diagrams and these sums are reduced. The final formula only contains finite sums.

The usefulness of the generating functions is illustrated in § 9 where an explicit formula for an arbitrary CRC is extracted and in § 10 where an example of recursion relations is given. Also in § 9 we prove that the symmetries of the $3j$, $6j$ and $9j$ coefficients induced by invertible linear transformations on the j of the coefficients, are the known symmetries of the coefficients.

2. Definition of the generating function Φ_G

In this section we define a number of notations, most of which are adopted from Bargmann (1962, see also Biedenharn and Van Dam 1965, pp 300–16), and the generating function Φ_G of a Jucys graph G.

We utilize Jucys graphs with only two types of free branches (co- and contravariant) and one kind of vertex. The diode symbols represent $2jm$ coefficients. For the sake of simplicity we shall assume most of the time that the graphs do not contain diode symbols.

A Jucys graph G is made up of a vertices, b free branches and c bound branches. Here G is considered as a structure, not assigning peculiar values to the j and m. For every vertex v, where three branches j_1, j_2 and j_3 meet, we define:

$$J_v = j_1 + j_2 + j_3, \qquad k_i = J_v - 2j_i \qquad (1 \leqslant i \leqslant 3).$$

We call k_1, k_2, k_3 the l indices of vertex v. The triangle condition (j_1, j_2, j_3) is equivalent to the condition: $k_i \in N$ $(1 \leqslant i \leqslant 3)$ (N is the set of non-negative integers). For every free branch (j, m) we define $\kappa = j + m$, $\lambda = j - m$ that we call the l indices of the free branch. The condition m projection of j is equivalent to $\kappa \in N$, $\lambda \in N$. We arrange the l indices of the vertices and free branches as an ordered set $L = (l_1, l_2, \ldots, l_{n_G})$ with

$n_G = 3a+2b$. When the j and m take all possible values compatible with triangle and projection conditions, L runs on a subset E_G of N^{n_G} (the l indices are not independent: to every bound branch j_1 corresponds the relation $k_2+k_3 = k'_2+k'_3$ $(=2j_1)$, to every free branch (j_1, m_1) the relation $\kappa_1+\lambda_1 = k_2+k_3\,(=2j_1)$). For every $L \in E_G$, corresponding to particular values of j and m, we denote the value of the CRC by G_L or by $G_{j_1 m_1 ..}$ and we define a normalization constant

$$N_L = \left(\frac{\Pi(J_v+1)!}{[L!]}\right)^{1/2}, \tag{1}$$

where the product runs over the vertices of G and $[L!] = l_1! l_2! \dots l_{n_G}!$.

We now define the generating function Φ_G of graph G as an entire function of $\tau = (\tau_1, \tau_2, \dots, \tau_{n_G}) \in C^{n_G}$ by

$$\Phi_G = \sum_{L \in E_G} N_L G_L \tau^{[L]}, \tag{2}$$

where $\tau^{[L]} = \tau_1^{l_1}\tau_2^{l_2} \dots \tau_{n_G}^{l_n}$. We say that τ_i is the variable corresponding to the l index l_i. Instead of τ_i, we shall often use the notation ξ_i or η_i corresponding respectively to l indices of the type κ or λ.

3. Elementary operations on graphs and generating functions

The graphs of the $3j$ coefficient $\left(\begin{smallmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{smallmatrix}\right)$ and of the $2jm$ coefficient $\delta_{jj'}\,\delta_{mm'}$ are represented on the left of figures 1 and 2. Their generating functions are known and will be given in § 6. The most general CRC is obtained from $3j$ and $2jm$ coefficients by carrying out a sequence of elementary operations: product, change of type of an index (co- and contravariant), permutation of j and summation on m. In terms of graphs these operations are: union of graphs, changes of the type of a free branch and sign of a vertex, connecting of two free branches. When these elementary operations (to which we add the change of direction of a free branch) are made on Jucys graphs, the generating functions transform in a definite way that we now describe in detail.

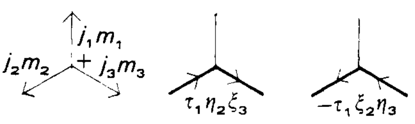

Figure 1. The graph, two open diagrams and their associated monomials for the $3j$ coefficient.

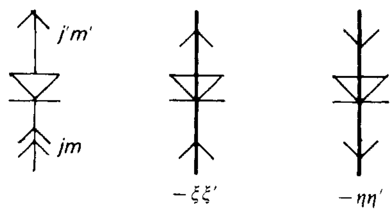

Figure 2. The graph, two open diagrams and their associated monomials for the $2jm$ coefficient.

3.1. *Product*

To the product of two CRC of graphs G' and G'' corresponds the union of the graphs $G = G' \cup G''$ and the product of the generating functions $\Phi_G = \Phi_{G'}\Phi_{G''}$.

3.2. *Types of a free branch*

CRC with co- and contravariant indices are described by graphs with two types of free branches (figure 3, where, as in the following, we represent part of the graph as a box). The generating function of G is denoted by $\Phi_G(\xi, \eta, \tau_B)$, where, as in the following, the variables τ_B correspond to the l indices L_B of part B of the graph and the first variables correspond to the l indices of part $G-B$ (here the variables ξ, η correspond to the l indices of the free branch κ, λ). Graphs G and G' of figure 3 are related by $G'_{jm\ldots} = (-1)^{j-m} G_{j-m\ldots}$ or in terms of l indices $G'_{\kappa\lambda L_B} = (-1)_{\lambda\kappa L_B}$. We then have:

$$\Phi_{G'}(\xi', \eta', \tau'_B) = \Phi_G(-\eta', \xi', \tau'_B).$$

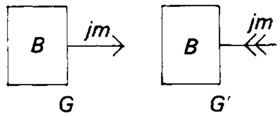

Figure 3. Change of the type of a free branch.

3.3. *Sign of a vertex*

At each vertex the branches are ordered but for a cyclic permutation. This order is indicated by a + or − sign. For the graphs in figure 4 we have

$$G'_{j_1 j_2 j_3\ldots} = (-1)^{j_1+j_2+j_3} G_{j_1 j_2 j_3\ldots}$$

or

$$G'_{k_1 k_2 k_3 L_B} = (-1)^{k_1+k_2+k_3} G_{k_1 k_2 k_3 L_B}$$

and

$$\Phi_{G'}(\tau'_1, \tau'_2, \tau'_3, \tau'_B) = \Phi_G(-\tau'_1, -\tau'_2, -\tau'_3, \tau'_B),$$

where the variables $\tau'_1, \tau'_2, \tau'_3$ correspond to the l indices k_1, k_2, k_3 of the vertex.

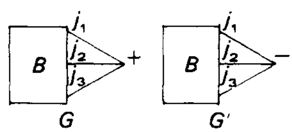

Figure 4. Change of the sign of a vertex.

3.4. *Contraction*

For the CRC $G'_{j'm'\,j''m''\ldots}$ (figure 5) the contraction on the co- and contravariant indices m' and m'' is made when $j' = j''$ by summing over $m' = m''$. Graph G corresponds to the definition:

$$G_{L_B} = \sum_{m'} G'_{j'm'\,j'm'\ldots} = \sum \delta_{\kappa'\kappa''}\,\delta_{\lambda'\lambda''}\,G'_{\kappa'\lambda'\kappa''\lambda'' L_B},$$

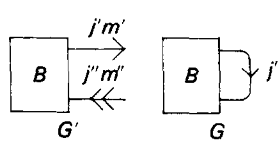

Figure 5. Contraction.

where the second sum is over $L' = (\kappa', \lambda', \kappa'', \lambda'', L_B) \in E_{G'}$ with given L_B. The corresponding graph G is also given in figure 5. Since $N_{L'} = (\kappa'!\,\lambda'!\,\kappa''!\,\lambda''!)^{-1/2} N_{L_B}$, the generating function of G is obtained from the series expansion of $\Phi_{G'}(\xi', \eta', \xi'', \eta'', \tau_B)$ by replacing $\xi'^{\kappa'}\,\eta'^{\lambda'}\,\xi''^{\kappa''}\,\eta''^{\lambda''}$ by $\kappa'!\,\lambda'!\,\delta_{\kappa'\kappa''}\,\delta_{\lambda'\lambda''}$. Remarking that

$$\kappa'!\,\lambda'!\,\delta_{\kappa'\kappa''}\,\delta_{\lambda'\lambda''} = \int \xi^{\kappa'}\,\bar{\eta}^{\lambda'}\,\xi^{\kappa''}\,\eta^{\lambda''}\,\mathrm{d}\mu_1(\xi)\,\mathrm{d}\mu_1(\eta)$$

(where for $z = x + \mathrm{i}y$, $\bar{z} = x - \mathrm{i}y$ $\mathrm{d}\mu_1(z) = \pi^{-1}\,\mathrm{e}^{-z\bar{z}}\,\mathrm{d}x\,\mathrm{d}y$ is integrated over R^2) the generating function Φ_G can be expressed as:

$$\Phi_G(\tau_B) = \int \Phi_{G'}(\bar{\xi}, \bar{\eta}, \xi, \eta, \tau_B)\,\mathrm{d}\mu_1(\xi)\,\mathrm{d}\mu_1(\eta).$$

3.5. *Direction of a bound branch*

From the definition of contraction, for two CRC represented by graphs G and G' (figure 6) differing by the direction of branch j_1 we have

$$G'_{j_1 j_2 j_3 \ldots} = (-1)^{2j_1} G_{j_1 j_2 j_3 \ldots}$$

or

$$G'_{k_1 k_2 k_3 L_B} = (-1)^{k_2 + k_3} G_{k_1 k_2 k_3 L_B}$$

and

$$\Phi_{G'}(\tau'_1, \tau'_2, \tau'_3, \tau'_B) = \Phi_G(\tau'_1, -\tau'_2, -\tau'_3, \tau'_B).$$

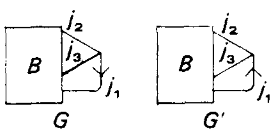

Figure 6. Change of the direction of a bound branch.

Likewise, changing the direction of the diode symbol in the $2jm$ coefficient (figure 2) gives a factor $(-1)^{2j}$ and a similar relation between generating functions.

4. Diagrams

This paragraph is devoted to the definition of certain graphs drawn on the Jucys graph. For a Jucys graph G we denote by $a, b \ldots$ the vertices, by $(ab)_s$ or $(ba)_s$ with $s = 1, \ldots n$ the n branches ($n \leqslant 3$) connecting a and b. A *passage at vertex b* is defined as the ordered set of two different branches $\phi = (ab)_s$ and $\phi' = (bc)_{s'}$ which are linked at vertex b. We represent it by $]\phi\,\phi'[$ or $]a\,b\,c[$, where to simplify notations subscripts s and s' are not expressed. Similarly, from now on, branch $(ab)_s$ will be denoted by (ab), omitting subscript s. Branch ϕ (or ϕ') and passage $]\phi\,\phi'[$ are said to be *connected*. A *path* is an ordered sequence of alternating branches and passages connected one to the following.

Example: $\phi = (ab)$, $]\phi\,\phi'[$, $\phi' = (bc)$, $]\phi'\phi''[$, $\phi'' = (cd)$.

This path is represented by $[a\,b\,c\,d]$ or $[\phi\,b\,c\,\phi'']$. The *direction* of the path on branch ϕ' is the one from b to c. The *extremities* of $[\phi\,b\,c\,\phi'']$ are branches ϕ and ϕ''. We use the notations $]a\,b\,c\,d]$ or $]\phi\,b\,c\,\phi'']$ to represent the path obtained from the preceding one by striking out branch ϕ, and similarly $[\phi\,b\,c\,\phi''[$, $]\phi\,b\,c\,\phi''[$ etc. Paths $[\phi\,b\ldots z\,\phi']$ and $[\phi'\,z\ldots b\,\phi]$ are said to be *reversed.* A *free path* is a path whose extremities are free branches. A *directed circuit* is an ordered cycle of alternating branches and passages connected one to the following.

Example: $\phi = (ab),]\phi\,\phi'[, \phi' = (bc),]\phi'\phi''[, \phi'' = (ca),]\phi'', \phi[$.

This circuit is represented by $(a\,b\,c)$ or $(b\,c\,a)$ or $(c\,a\,b)$ (this notation causes no confusion with that for branches). A π *circuit* is a circuit that can be separated into n identical circuits $(n > 1)$ (example: $(a\,b\,a\,b)$). A circuit which is not a π circuit is called *non-π*. Circuits $(a\,b\,c)$ and $(c\,b\,a)$ are said to be *reversed*; they both represent the same *non-directed* circuit. A set D of paths $P_1, P_2\ldots$ and circuits $C_1, C_2\ldots$ (or: paths and non-directed circuits) is called a *directed* (or: *non-directed*) *diagram* and it will be represented graphically on graph G by thick lines with directions indicated if necessary by arrows. We say that diagram D is *composed* of paths $P_1, P_2\ldots$ and circuits $C_1, C_2\ldots$. The passages and branches of a diagram D, each passage or branch being counted as many times as it appears in D are called the *elements of D*. Their set will be denoted by $\mathscr{B}(D)$. A diagram the elements of which are not repeated is said to be *simple.* A *closed diagram* is a simple diagram composed of n non-directed circuits $(n \geqslant 1)$. An *open diagram* is a simple diagram composed of one free path and of n non-directed circuits $(n \geqslant 0)$.

We define the following sets of diagrams of G, V being a set of branches of G:

$\mathscr{D} = \{\text{diagrams of } G\}$;
$\mathscr{D}_V = \{D \in \mathscr{D}: \text{some branches of } V \text{ appear in } \mathscr{B}(D)\}$; $\overline{\mathscr{D}}_V = \mathscr{D} - \mathscr{D}_V$;
$\mathscr{R}_V = \{D \in \mathscr{D}: \text{every branch of } V \text{ appears exactly once in } \mathscr{B}(D)\}$;
$K_G = \{\text{closed diagrams}\}$;
$\Omega_G = \{\text{open diagrams}\}$;
$H_G = \{\text{non-directed circuits}\}$;
$H'_G = \{\text{non-directed, non-}\pi\text{ circuits}\}$;
$F_G = \{\text{free paths}\}$;
$F'_G = \{\text{free paths with different extremities}\}$;
$Q_G = \{\text{simple free paths}\}$.

The sets K_G, Ω_G and Q_G are finite. The other sets are generally infinite.

5. The function $D \to M(D)$

In the following paragraphs, various algebraic expressions will be interpreted in terms of diagrams by means of a monomial $M(D)$ defined for every diagram D. If D is a directed diagram we construct the monomial $M(D)$ by multiplying the following factors which are associated with various parts of D:

(*a*) For every free branch jm belonging to $\mathscr{B}(D)$, of l indices κ, λ and corresponding variables ξ, η, we associate a factor given in figure 7 and depending on the type of the branch and on the direction of the diagram on the branch.

(*b*) For every passage $]j_1\,j_2[$ belonging to $\mathscr{B}(D)$ (vertex v is connected to branches j_1, j_2, j_3; the l indices of v are k_1, k_2, k_3 and the corresponding variables τ_1, τ_2, τ_3)

(see figure 8) we associate the factor τ_3 (or $-\tau_3$) if the order of the branches at the vertex is j_1, j_2, j_3 (or j_2, j_1, j_3).

(*c*) For every bound branch belonging to $\mathscr{B}(D)$ (figure 9) we associate a factor -1 (or $+1$) if the directions of the branch in D and G are identical (or opposite).

(*d*) Every time diagram D passes backwards through a diode symbol (figure 10) we associate a factor -1.

(*e*) For every circuit in D we associate a factor -1.

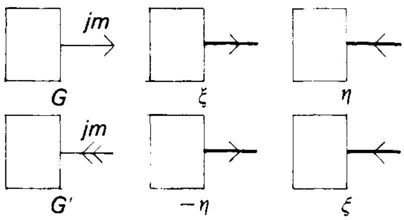

Figure 7. Computation of $M(D)$.

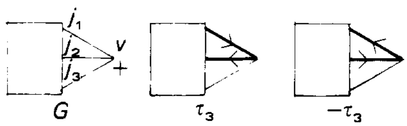

Figure 8. Computation of $M(D)$.

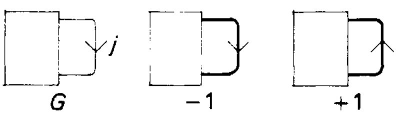

Figure 9. Computation of $M(D)$.

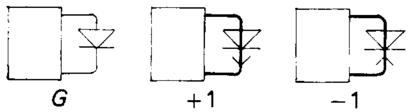

Figure 10. Computation of $M(D)$.

Remarks

(i) $M(D) = M(D')$ if D' is the diagram obtained by reversing the direction of some circuits in D. Thus $M(D)$ is also defined for non-directed diagrams.

(ii) $M(D'') = -M(D)$ if D'' is obtained by reversing the direction of a path of D whose extremities are bound branches.

(iii) The monomial $M(D)$ is transformed in the same way as the generating functions in the elementary operations in §§ 3.2, 3.3 and 3.5 (diagram D remaining unchanged).

(iv) If diagram D is composed of diagrams D_1 and D_2, then $M(D) = M(D_1)M(D_2)$. If diagrams D and D' are composed of n and n' circuits only and have the same set of elements, then $M(D) = (-1)^{n-n'}M(D')$.

(v) In the following paragraphs, there occur infinite sums $\Sigma\, M(D)$ and products $\Pi(1+M(D))$ over sets of diagrams like H_G, H'_G and F_G. Since the number of diagrams of these sets that have p elements is an $O(\mu^p)$ for $p \to \infty$, the sums and products are absolutely convergent for sufficiently small τ.

We also define the functions $D \to \epsilon(D)$ into $\{-1, +1\}$, $D \to l_i(D)$ into N $(1 \leqslant i \leqslant n_G)$ and $D \to L(D) = (l_1(D), l_2(D), \ldots, l_{n_G}(D))$ into N^{n_G} by setting $M(D) = \epsilon(D)\tau^{[L(D)]}$. We call $L(D)$ the l indices of diagram D. In § 9 it will be shown that the l indices of a diagram composed of circuits and free paths can be interpreted as the l indices of a CRC.

6. The generating functions for the 3*j* and 2*jm* coefficients

For the 3*j* (figure 1) we denote the l indices by k_1, k_2, k_3 (vertex), κ_i, λ_i (branch $j_i m_i$ $1 \leqslant i \leqslant 3$) and the corresponding variables by $\tau_1, \tau_2, \tau_3, \xi_i, \eta_i$ $(1 \leqslant i \leqslant 3)$. There are six open diagrams on the $3j$, two of which are drawn on figure 1. We have

$$\sum_{T\in\Omega_{3j}} M(T) = -\begin{vmatrix} \tau_1 & \tau_2 & \tau_3 \\ \xi_1 & \xi_2 & \xi_3 \\ \eta_1 & \eta_2 & \eta_3 \end{vmatrix},$$

so that equation (3.21) in Bargmann (1962, see also Biedenharn and Van Dam 1965, p 310) for the generating function of the $3j$ coefficient can be transcribed in terms of diagrams as:

$$\Phi_{3j} = \exp\left(-\sum_{T\in\Omega_{3j}} M(T)\right). \tag{3a}$$

The two open diagrams of the $2jm$ are given in figure 2, from which:

$$\Phi_{2jm} = \exp(\xi\xi'+\eta\eta') = \exp\left(-\sum_{T\in\Omega_{2jm}} M(T)\right). \tag{3b}$$

7. Algebraic expression of the generating function Φ_G

In this section we obtain an algebraic expression for the generating function Φ_G by starting from the generating functions for the $3j$ and $2jm$ coefficients and carrying out a sequence of elementary operations (§ 3). We represent (figure 11) graph G with its n bound branches i ($1 \leqslant i \leqslant n$) and p free branches i' ($1 \leqslant i \leqslant p$) of arbitrary types (not represented), so that the box B contains only the vertices and diode symbols of the graph.

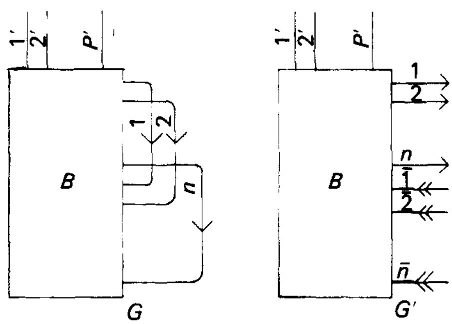

Figure 11. Obtainment of graph G by contractions on G'.

G is obtained from G' (figure 11) by n contractions, ie by connecting the free branches i and $\bar{i}$ ($1 \leqslant i \leqslant n$) of graph G' (elementary operation of § 3.4). Since G' is obtained from $3j$ and $2jm$ coefficients by the elementary operations of §§ 3.1, 3.2 and 3.3, we get using equations ($3a$, b) and remark (iii) of § 5:

$$\Phi_{G'} = \exp\left(-\sum_{T\in\Omega_{G'}} M(T)\right). \tag{4}$$

There is either zero or one open path (denoted by $[i\ k]$) starting from a given branch i and ending at another given branch k in graph G'. We recall that the paths obtained by removing branch i and/or k from $[i\,k]$ are denoted by $]i\,k[$, $[i\,k[$ and $]i\,k]$. If path $[i\,k]$ does not exist we put $M([i\,k]) = 0$. The branches i and k are taken from the sets $\beta = \{1, 2, 3 \ldots n\}$, $\bar\beta = \{\bar 1, \bar 2, \bar 3 \ldots \bar n\}$ and $\alpha = \{1', 2' \ldots p'\}$. We put for $i \in \beta \cup \bar\beta$, $j \in \beta \cup \bar\beta$, $k \in \alpha$, $l \in \alpha$

$$a_{ij} = M(]i\,j[), \qquad b_{il} = M(]i\,l]), \qquad v_i = \sum_{l\in\alpha} b_{il}, \qquad c_{kj} = M([k\,j[),$$

$$w_j = \sum_{k\in\alpha} c_{kj}, \qquad d_{kl} = M([k\,l]) \qquad \text{and} \qquad |d| = \sum_{\substack{k\in\alpha\\ l\in\alpha}} d_{kl}.$$

Let

$$t = (\xi_1, \xi_2 \ldots \xi_n, -\eta_{\bar 1}, -\eta_{\bar 2} \ldots -\eta_{\bar n}), \qquad t' = (\xi_{\bar 1}, \xi_{\bar 2} \ldots \xi_{\bar n}, -\eta_1, -\eta_2 \ldots -\eta_n)$$

and

$$z = (\eta_1, \eta_2 \ldots \eta_n, \xi_{\bar 1}, \xi_{\bar 2} \ldots \xi_{\bar n})$$

be vectors of the space C^{2n}, where ξ_i, η_i are the variables corresponding to the l indices of the free branch $i \in \beta \cup \bar\beta$; $u\,.\,u' = \Sigma_{i=1}^{2n} u_i u_i'$ is the scalar product of $u = (u_1 \ldots u_{2n}) \in C^{2n}$ and $u' = (u_1' \ldots u_{2n}') \in C^{2n}$; at is the vector with component i given by $(at)_i = \Sigma_{j=1}^{2n} a_{ij} t_j$. We have $z = \tilde{f} t'$ with

$$f = \left(\begin{array}{c|c} 0 & 1_n \\ \hline -1_n & 0 \end{array}\right)$$

and

$$\sum_{T\in\Omega_G'} M(T) = z.at + z.v + w.t + |d| = t'.fat + t'.fv + w.t + |d|.$$

According to § 3.4 the generating function Φ_G is obtained from equation (4) by:

$$\Phi_G = \int \exp(-\bar t.fat - \bar t.fv - w.t - |d|)\, d\mu_{2n}(t) \tag{5}$$

where $d\mu_{2n}(t) = \Pi_{i=1}^{2n} d\mu_1(t_i)$. The integral can be computed, for fa sufficiently small, by the method of the appendix of Bargmann (1962, see also Biedenharn and Van Dam 1965, pp 315–6):

$$\Phi_G = [\det(1 + fa)]^{-1} \exp[w\,.\,(1 + fa)^{-1} fv - |d|]. \tag{6}$$

8. Evaluation of the generating function in terms of closed and open diagrams

Most of the quantities in equation (6) were interpreted (§ 7) in terms of diagrams of G'. In this section interpretations in terms of diagrams of G are obtained. We first transform the determinant (§ 8.1) in equation (6), expressing it as a generally infinite product over the non-directed non-π circuits of G. With the aid of a formula in the appendix it is reduced to an expression containing only a finite sum over the closed diagrams of G. Similarly (§ 8.2) the exponent in equation (6) is shown to be minus the sum of $M(T)$ over the free paths T of G. It is then expressed in terms of finite sums only over the closed and open diagrams of G. Some examples are considered in § 8.3.

8.1. Transformation of determinant in equation (6)

Letting $x = -fa$ we have

$$\ln \det(1-x) = \mathrm{Tr}[\ln(1-x)] = -\left(\sum_{i} x_{ii} + \tfrac{1}{2}\sum_{ij} x_{ij}x_{ji} + \tfrac{1}{3}\sum_{ijk} x_{ij}x_{jk}x_{ki} + \ldots\right) \tag{7}$$

where the sums are made on $i \in \beta \cup \bar{\beta}, j \in \beta \cup \bar{\beta}, k \in \beta \cup \bar{\beta} \ldots$. We now interpret equation (7) in terms of diagrams of G. For $i \in \beta, j \in \beta \cup \bar{\beta}$, we have $x_{ij} = -a_{ij}$, $x_{ij} = a_{ij}$, hence $x_{ij} = M(P_{ij})$, $x_{ij} = M(P_{ij})$, where P_{ij} is the path of G composed of branch i and path $]i\,j[$ (as defined in § 7) and P_{ij} is the path of G composed of branch i and path $]i\,j[$. Here the paths $]\bar{i}\,j[$ and $]i\,j[$ of G' which are contained in box B (figure 11) are interpreted as paths of G.

Every term of equation (7) is then of the form:

$$-x_{ij}x_{jk} \ldots x_{mi} = -M(P_{ij})M(P_{jk}) \ldots M(P_{mi}) = M(C)$$

where C is the circuit $P_{ij}, P_{jk} \ldots P_{mi}$. Conversely, let C be a non-directed circuit of q branches; if C can be separated into n identical non-π circuits S ($n \geqslant 1$), we set $\pi(C) = n$. Then $M(C) = -[-M(S)]^{\pi(C)}$. $M(C)$ appears $2q/\pi(C)$ times in equation (7). So equation (7) is transcribed as a sum on the non-directed circuits, and then on the non-directed non-π circuits:

$$\ln \det(1-x) = 2\sum_{C \in H_G} M(C)/\pi(C) = 2\sum_{S \in H'_G} [M(S) - \tfrac{1}{2}(M(S))^2 + \tfrac{1}{3}(M(S))^3 - \ldots]$$

$$= 2\sum_{S \in H'_G} \ln(1 + M(S)).$$

Hence

$$\det(1-x) = \left(\prod_{S \in H'_G} (1 + M(S))\right)^2. \tag{8}$$

From equation (A.5) (with $V = \varnothing$ (null set)) in the appendix the product in equation (8) is expressed as a finite sum over the closed diagrams:

$$\det(1+fa) = \left(1 + \sum_{D \in K_G} M(D)\right)^2. \tag{9}$$

8.2. Transformation of the exponent in equation (6)

Expanding $(1-x)^{-1}$ in powers of x we get:

$$w\,.\,(1-x)^{-1}fv = \sum_{kil} c_{ki}(fb)_{il} + \sum_{kijl} c_{ki}x_{ij}(fb)_{jl} + \sum_{kijml} c_{ki}x_{ij}x_{jm}(fb)_{ml} + \ldots \tag{10}$$

where the sums are on $k \in \alpha, l \in \alpha, i \in \beta \cup \bar{\beta}, j \in \beta \cup \bar{\beta}, m \in \beta \cup \bar{\beta}, \ldots$. For $i \in \beta, l \in \alpha$ we have $(fb)_{il} = b_{il}$, $(fb)_{il} = -b_{il}$ and then $(fb)_{il} = -M(Q_{il})$, $(fb)_{il} = -M(Q_{il})$, where Q_{il} is the path of G composed of branch i and path $]\bar{i}\,l]$ of G' (now interpreted as of G) and Q_{il} is the path of G composed of branch i and path $]i\,l]$.

Every term of equation (10) is then of the form:

$$c_{ki}x_{ij} \ldots x_{mn}(fb)_{nl} = -M([ki[)M(P_{ij}) \ldots M(P_{mn})M(Q_{nl}) = -M(T)$$

where T is the free path of G: $[ki[, P_{ij} \ldots P_{mn}, Q_{nl}$. Conversely to every free path T of

G there corresponds one term in equation (10), if T passes over a bound branch of G, or one term in $-|d|$ if T does not pass over a bound branch. The exponent in equation (6) is thus expressed as a sum over the free paths:

$$w\,.\,(1+fa)^{-1}fv-|d| = -\sum_{T\in F_G} M(T). \tag{11}$$

If T and T' are reversed free paths that start and end on the same free branch, then by § 5, remark (ii) $M(T) = -M(T')$ so that the sum in equation (11) can be limited to $T \in F'_G$.

We now define a procedure that decomposes every free path into a simple free path and circuits. It will then be possible to simplify equation (11). We need some notations.

Let $t = (\phi_1 u_1 u_2 \dots u_p \phi_{p+1}] \in Q_G$ be a simple free path of p vertices. We introduce the following two sets of labels occurring in t: $\omega_t = \{1, 2, \dots, p-1\}$, $\omega'_t = \omega_t \cup \{p\}$ and we put $\phi_i = (u_{i-1}u_i) \in \mathscr{B}(t)$ for $2 \leqslant i \leqslant p$. For $k \in \omega'_t$ let $S_{t,k}$ be the set of paths of the form $[u_k \dots z u_k \phi_{k+1}[$ that do not contain branches ϕ_i $(1 \leqslant i \leqslant k)$ and $S'_{t,k}$ the set of paths of $S_{t,k}$ that begin with branch ϕ_{k+1} (so $S'_{t,p} = \varnothing$). For $k \in \omega_t$ let $R_{t,k}$ be the set of non-directed circuits that pass over branch ϕ_{k+1} exactly once and that do not contain branches ϕ_i $(1 \leqslant i \leqslant k)$.

The procedure (P) is defined as follows. Let $T = [\Psi_1 a_1 a_2 \dots a_n \Psi_{n+1}] \in F'_G$ be a free path of n vertices and whose extremities Ψ_1 and Ψ_{n+1} are different free branches. We put $\Psi_k = (a_{k-1}a_k) \in \mathscr{B}(T)$ $(2 \leqslant k \leqslant n)$. If T is not simple let v be the first vertex in the sequence $a_1 a_2 \dots a_n$ that appears at least twice, k and k' the smallest and greatest numbers such that $a_k = a_{k'} = v$. Put $T' = [\Psi_1 a_1 \dots a_k a_{k'+1} \dots \Psi_{n+1}] \in F'_G$ and

$$T'' = [a_k a_{k+1} \dots a_{k'} \Psi_{k'+1}[.$$

Then T can be reconstructed by inserting path T'' into path T' at vertex a_k. Notice that T'' does not pass on branches Ψ_i $(1 \leqslant i \leqslant k)$ and that $\mathscr{B}(T) \neq \mathscr{B}(T') \cup \mathscr{B}(T'')$ in general. Repeating on T' the same process as on T, and iterating we obtain uniquely determined $t \in Q_G$, $\omega' \subset \omega'_t$ and for $k \in \omega'$ paths $T_k \in S_{t,k}$.

Conversely let $t \in Q_G$, $\omega' \subset \omega'_t$ and for $k \in \omega'$ paths $T_k \in S_{t,k}$, then by inserting, for every k in ω', path T_k at vertex u_k of t we obtain a free path $T \in F'_G$, which when decomposed by P gives back the same t, ω' and T_k. So the correspondence:

$$T \in F'_G \rightleftarrows \begin{cases} t \in Q_G \\ \omega' \subset \omega'_t \\ T_k \in S_{t,k}\ (k \in \omega') \end{cases} \tag{P}$$

is one-to-one.

Let F''_G be the set of paths $T \in F'_G$ such that in P every $T_k \in S'_{t,k}$.

If in P, for a given k, $T_k \in S_{t,k} - S'_{t,k}$ then it is of the form $T_k = [u_k Z y_1 y_2 \dots y_q Z u_k \phi_{k+1}[$, with identical first and last branches $(u_k Z)$ and $T'_k = [u_k Z y_q \dots y_2 y_1 Z u_k \phi_{k+1}[$ is a path of $S_{t,k} - S'_{t,k}$ that is different from T_k. By replacing T_k by T'_k in P we obtain T' instead of T, and from § 5, remark (ii) $M(T') = -M(T)$. Thus $\Sigma_{T\in F_G - F''_G} M(T) = 0$.

If $T \in F''_G$ we pursue the decomposition further. For a given k, let $r(k)$ be the number of times branch ϕ_{k+1} appears in $T_k \in S'_{t,k}$ $(r(k) \geqslant 1)$. T_k can be decomposed in an ordered sequence $C^k_1, C^k_2, \dots, C^k_{r(k)}$ of circuits of $R_{t,k}$. Conversely let $C^k_1, C^k_2, \dots, C^k_{r(k)}$ be an ordered sequence of arbitrary length $r(k) \geqslant 1$ of circuits of $R_{t,k}$. If $C^k_i = (u_k u_{k+1} \dots Z)$ we define path $P_i = [u_k u_{k+1} \dots Z u_k \phi_{k+1}[$. Then by joining paths $P_1 P_2 \dots P_{r(k)}$ we obtain a path T_k of $S'_{t,k}$.

In summary, the correspondence

$$T \in F''_G \rightleftarrows \begin{cases} t \in Q_G \\ \omega' \subset \omega_t \\ C_i^k \in R_{t,k} \qquad 1 \leqslant i \leqslant r(k), \qquad k \in \omega' \end{cases}$$

is one-to-one, and:

$$M(T) = M(t) \prod_{k \in \omega'} \prod_{i=1}^{r(k)} (-M(C_i^k)).$$

Putting $X_{t,k} = \Sigma_{C \in R_{t,k}} M(C)$, we have then:

$$\sum_{T \in F''_G} M(T) = \sum_{t \in Q_G} M(t) \prod_{k \in \omega_t} (1 - X_{t,k} + X_{t,k}^2 - X_{t,k}^3 + \dots)$$

from which:

$$\sum_{T \in F_G} M(T) = \sum_{t \in Q_G} M(t) \prod_{k \in \omega_t} (1 + X_{t,k})^{-1}. \tag{12}$$

From equation (A.3) in the appendix, since $R_{t,k} = H'_G \cap \mathscr{R}_W \cap \overline{\mathscr{D}}_V$, with

$$V = \{\phi_2, \dots \phi_k\} \quad (V = \varnothing \text{ if } k = 1) \qquad \text{and} \quad W = \{\phi_{k+1}\}$$

and putting $U_{t,k} = H'_G \cap \mathscr{D}_W \cap \overline{\mathscr{D}}_V$:

$$1 + X_{t,k} = \prod_{C \in U_{t,k}} (1 + M(C)).$$

The sets $(U_{t,k})_{k \in \omega_t}$ form a partition of $H'_G \cap \mathscr{D}_{W_t}$ where W_t is the set of the bound branches of t. Equation (12) becomes:

$$\sum_{T \in F_G} M(T) = \sum_{t \in Q_G} M(t) \prod_{C \in H'_G \cap \mathscr{D}_{W_t}} (1 + M(C))^{-1}$$
$$= \left(\prod_{C \in H'_G} (1 + M(C)) \right)^{-1} \sum_{t \in Q_G} M(t) \prod_{C \in H'_G \cap \overline{\mathscr{D}}_{W_t}} (1 + M(C)).$$

And by equation (A.5) (used with $V = \varnothing$ and $V = W_t$) in the appendix:

$$\sum_{T \in F_G} M(T) = \left(1 + \sum_{D \in K_G} M(D) \right)^{-1} \sum_{t \in Q_G} M(t) \left(1 + \sum_{D \in K_G \cap \overline{\mathscr{D}}_{W_t}} M(D) \right)$$
$$= \left(1 + \sum_{D \in K_G} M(D) \right)^{-1} \sum_{T \in \Omega_G} M(T). \tag{13}$$

From equations (6), (9), (11) and (13) we obtain the final expression for the generating function:

$$\begin{aligned} \Phi_G &= A^2 \exp\left(-A \sum_{T \in \Omega_G} M(T) \right) \\ A &= \left(1 + \sum_{D \in K_G} M(D) \right)^{-1}. \end{aligned} \tag{14}$$

The generating function Φ_G is thus expressed in terms of the finite sums of $M(D)$ over the closed and open diagrams of G. In order to write Φ_G explicitly one has to determine the closed and open diagrams of graph G and compute the corresponding monomials $M(D)$ by the rules in § 5.

8.3. Examples

For a $3nj$ coefficient, equation (14) simplifies to

$$\Phi_{3nj} = \left(1 + \sum_{D \in K_{3nj}} M(D)\right)^{-2}.$$

The $6j$ coefficient $\left\{\begin{smallmatrix} j_{01} & j_{02} & j_{03} \\ j_{23} & j_{13} & j_{12} \end{smallmatrix}\right\}$ is represented by a tetrahedron (figure 12) with vertices V_i $(0 \leqslant i \leqslant 3)$; branch j_{ik} $(i < k)$ connects V_i and V_k; we label the l indices of vertex V_i by k_{ij} $(0 \leqslant j \leqslant 3, j \neq i)$ and the corresponding twelve variables by τ_{ij}. There are seven

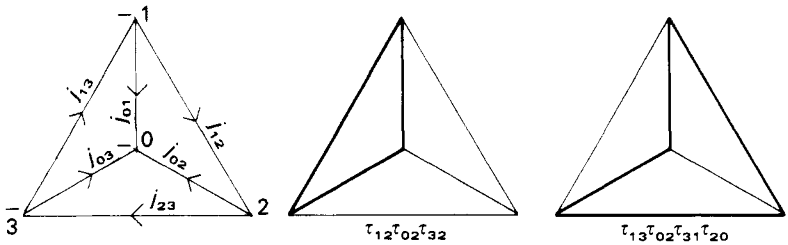

Figure 12. The graph, two closed diagrams and their associated monomials for the $6j$ coefficient.

closed diagrams (which are simple circuits): four circuits with three branches and three circuits with four branches; one of each kind is drawn in figure 12, and the corresponding $M(C)$ is given. From equation (14):

$$\Phi_{6j} = (1 + \tau_{10}\tau_{20}\tau_{30} + \tau_{01}\tau_{31}\tau_{21} + \tau_{32}\tau_{02}\tau_{12} + \tau_{23}\tau_{13}\tau_{03} + \tau_{01}\tau_{10}\tau_{23}\tau_{32} + \tau_{02}\tau_{20}\tau_{13}\tau_{31} + \tau_{03}\tau_{30}\tau_{12}\tau_{21})^{-2}$$

which is equation (4.15) of Bargmann (1962, see also Biedenharn and Van Dam 1965, p 313).

The $9j$ coefficient

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \\ j_7 & j_8 & j_9 \end{Bmatrix}$$

is represented by a cartwheel diagram (figure 13) with vertices $V_1, V_2, V_3, V_{1'}, V_{2'}, V_{3'}$. The l indices of vertex V_i (or: $V_{i'}$) are labelled k_{ij} (or: k'_{ij}) $(1 \leqslant j \leqslant 3)$, the corresponding variables by τ_{ij} (or: τ'_{ij}). There are fifteen closed diagrams (which are simple circuits): nine circuits D_a $(1 \leqslant a \leqslant 9)$ with four branches and six circuits $D_{a'}$ $(1 \leqslant a \leqslant 6)$ with

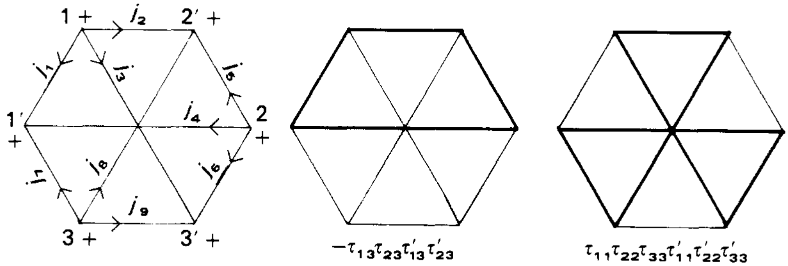

Figure 13. The graph, two closed diagrams and their associated monomials for the $9j$ coefficient.

six branches. We describe these closed diagrams (table 1) by their l indices $L(D_a)$ and the value of $\epsilon(D_a)$ (cf § 5). We also give the corresponding values of j_i. As will be clear from § 9 $-\epsilon(D_a)/2 = \frac{1}{2}$ $(1 \leqslant a \leqslant 9)$ or $-\epsilon(D_{a'})/4$ $(1 \leqslant a' \leqslant 6)$ is the value of the $9j$ coefficient with the indicated values of j_i. In figure 13 are drawn diagrams D_1 and $D_{1'}$. The generating function can be written down at once from table 1, thus reproducing the result of Wu (1972), but this will be omitted.

Table 1. Closed diagrams D_a ($a = 1, 2, \ldots, 9, 1', 2', \ldots, 6'$) of the $9j$ coefficient (figure 13): j indices, l indices (corresponding to vertices V_i), $\epsilon(D_a)$. Null values are omitted.

	$2j_i$									k_{ij}									k'_{ij}									
a	1	2	3	4	5	6	7	8	9	V_1			V_2			V_3			$V_{1'}$			$V_{2'}$			$V_{3'}$			ϵ
1	1	1		1	1							1			1						1			1				
2	1	1					1	1				1						1		1			1					
3				1	1		1								1			1	1			1						
4	1		1	1		1					1			1							1						1	
5	1		1				1		1		1						1			1						1		−1
6				1		1	1		1					1			1		1						1			
7		1	1		1	1				1			1											1			1	
8		1	1					1	1	1						1							1			1		
9					1	1		1	1				1			1						1			1			
1′		1	1	1		1	1	1		1				1				1	1				1				1	1
2′	1		1		1	1	1	1			1		1					1		1		1					1	−1
3′		1	1	1	1		1		1	1					1		1		1					1		1		−1
4′	1		1	1	1			1	1		1				1	1					1	1				1		1
5′	1	1			1	1	1		1			1	1				1			1				1	1			1
6′	1	1		1		1		1	1			1		1		1					1		1		1			−1

Huang and Wu (1974) have computed the generating function of the $12j$ coefficient represented in figure 14. Their result can be obtained from equation (14) in which there are 31 closed diagrams: 29 simple circuits and two diagrams composed of two circuits (one composite diagram D is represented (figure 14), and $M(D)$ is given in the notation of Huang and Wu 1974).

The graph G' for an njm coefficient is a tree in which there are no circuits. The generating function is then expressed by the same formula as for the $3j$ coefficient (equation (4)).

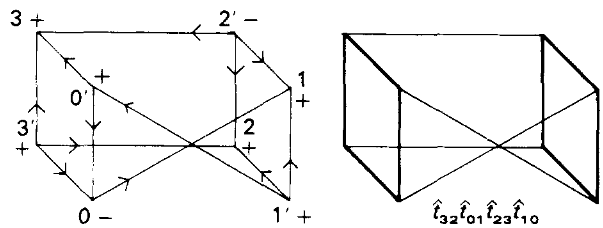

Figure 14. The graph, a closed diagram and its associated monomial for the $12j$ coefficient.

9. Explicit expression for the CRC

From the generating function, equation (14), we now obtain a general expression for any CRC. Symmetries are then briefly considered.

For a Jucys graph G let us denote respectively by T_i $(1 \leqslant i \leqslant p)$ and C_j $(1 \leqslant j \leqslant q)$ the open and closed diagrams. Expanding equation (14) we get

$$\Phi_G = \sum_{\alpha,\beta} \frac{(|\alpha|+|\beta|+1)!}{(|\alpha|+1)![\alpha!][\beta!]} \left(\prod_{i=1}^{p} (-M(T_i))^{\alpha_i} \prod_{j=1}^{q} (-M(C_j))^{\beta_j} \right), \tag{15}$$

where the sum is over $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p) \in N^p$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_q) \in N^q$; we have put $|\alpha| = \Sigma_{i=1}^{p} \alpha_i$, $|\beta| = \Sigma_{j=1}^{q} \beta_j$. By comparing equation (15) and equation (2) we obtain the expression:

$$G_L = N_L^{-1} \sum_{\alpha,\beta} \frac{(|\alpha|+|\beta|+1)!}{(|\alpha|+1)![\alpha!][\beta!]} \left(\prod_{i=1}^{p} (-\epsilon(T_i))^{\alpha_i} \prod_{j=1}^{q} (-\epsilon(C_j))^{\beta_j} \right), \tag{16}$$

where the summation is over the $\alpha \in N^p$, $\beta \in N^q$ such that

$$L = \sum_{i=1}^{p} \alpha_i L(T_i) + \sum_{j=1}^{q} \beta_j L(C_j) \tag{17}$$

and N_L as defined in equation (1).

The sum in equation (16) can be interpreted as being over all the possible ways of constructing diagrams from open and closed diagrams that have the same l indices as the calculated CRC.

For the $3j$ and $6j$ coefficients equation (16) can be written in terms of a summation on one integer and yields Racah's formulae.

To each closed or open diagram D $(=T_i$ or $C_j)$ with s vertices, t circuits $(t \geqslant 0)$ and r path $(r = 0$ or $1)$ there corresponds a CRC whose l indices are $L(D)$. The value of this CRC can be computed from equation (16) and is $(-1)^r 2^{-s/2}(-2)^t \epsilon(D)$. The $\epsilon(T_i)$ and $\epsilon(C_j)$ are thus interpreted in terms of the CRC whose l indices are the l indices of D.

9.1. Symmetry

Let E'_G be the set of $L \in E_G$ that satisfy polygonal conditions; we denote by E_i $(1 \leqslant i \leqslant r)$ the simple non-directed circuits and the simple free paths of G; we put $e_i = L(E_i) \in E'_G$. The set E'_G has the following properties:

(i) if $L_i \in E'_G$, $n_i \in N$ $(1 \leqslant i \leqslant h)$ then $\Sigma_{i=1}^{h} n_i L_i \in E'_G$;

(ii) if $L \in E'_G$ there exist $a_i \in N$ $(1 \leqslant i \leqslant r)$ such that $L = \Sigma_{i=1}^{r} a_i e_i$; moreover if $L = e_j$ the a_i are unique and $a_i = \delta_{ij}$.

Property (i) expresses the fact that E'_G is closed by addition. If $G_L \neq 0$ property (ii) stems from the above interpretation of equation (16), since the l indices L (which belongs to E'_G) can be written as in equation (17). In fact it is possible to prove property (ii) not assuming $G_L \neq 0$, but this will be omitted. The last part of property (ii) expresses that the e_j are extremal elements in E'_G.

We define a symmetry to be a function ϕ: $E'_G \to E'_G$ such that $G_{\phi(L)} = X_L G_L$ where X_L is a simple algebraic expression. Let us look for symmetries such that ϕ is an invertible linear function. Then ϕ is a permutation of $(e_i)_{1 \leqslant i \leqslant r}$. Indeed putting

$$\phi(e_i) = \sum_{j=1}^{r} a_{ij} e_j \qquad \text{and} \qquad \phi^{-1}(e_i) = \sum_{j=1}^{r} b_{ij} e_j$$

(the a_{ij} and b_{ij} not necessarily uniquely determined) we have

$$e_i = \sum_{jk} b_{ij} a_{jk} e_k$$

and from property (ii) of E'_G we obtain if $b_{ij} \neq 0$, $a_{jk} = \delta_{ki}/b_{ij}$ for $1 \leqslant k \leqslant r$. Since $a_{ji} \in N$ and $b_{ij} \in N$, $a_{ji} = 1/b_{ij}$ implies $a_{ji} = 1$, and $\phi(e_j) = e_i$.

We examine the case of the $9j$ coefficient. The E_i are the fifteen closed diagrams D_a described on table 1. We put $d_a = L(D_a)$, where a runs over $A = \{1, 2 \ldots 9\}$ and $B = \{1', 2' \ldots 6'\}$. $s = \frac{1}{4}\Sigma_{a \in A \cup B} d_a \in E'_G$ is an invariant by ϕ. The only sums of three elements d_a that add up to s are:

$$d_{1'} + d_{4'} + d_{5'} = d_{2'} + d_{3'} + d_{6'} = s. \tag{18}$$

So ϕ is a permutation of $(d_i)_{i \in B}$. We have $d_{1'} = s - d_1 - d_5 - d_9$, $d_{2'} = s - d_1 - d_6 - d_8$ so $d_{1'} + d_{2'} = 2s - 2d_1 - d_5 - d_6 - d_8 - d_9$ and in general for $a \in B$, $b \in B$, $a \neq b$: $d_a + d_b = 2s - 2d_c - d_d - d_e - d_f - d_g$ where c, d, e, f, g are different indices of A, this expression being unique. From this it follows that ϕ is determined by the $\phi(d_i)_{i \in B}$: for example if $\phi(d_{1'}) = d_a$, $\phi(d_{2'}) = d_b$ then $\phi(d_1) = d_c$. Moreover, equation (18) is invariant by ϕ. This leaves for ϕ $6 \times 6 \times 2 = 72$ choices which are the known symmetries of the $9j$ coefficient.

For the $3j$ coefficient the six extremal elements of E'_G are linked only by a relation like equation (18). The same method yields the $6 \times 6 \times 2 = 72$ Regge (1958) symmetries (see also Biedenharn and Van Dam 1965, pp 296–7).

For the $6j$ the seven extremal elements of E'_G are linked only by relation:

$$e_1 + e_2 + e_3 = e_4 + e_5 + e_6 + e_7$$

and the same method gives the $6 \times 24 = 144$ Regge (1959) symmetries (see also Biedenharn and Van Dam 1965, pp 298–9).

We have thus proved that the only invertible linear symmetries of the $3j$, $6j$ and $9j$ coefficients are the known symmetries.

10. Recursion relations

By differentiating Φ_G, many relations between G_L can be obtained. We give an example when G is a $3nj$ coefficient. Since

$$\tau_i \frac{\partial}{\partial \tau_i}\left(\sum_{C \in K_G} M(C)\right) = \sum_{C \in K_i} M(C),$$

where K_i is the set of closed diagrams D whose l index $l_i(D)$ is different from zero (and then equal to 1), we get:

$$\left[\left(\sum_{C \in K_i} M(C)\right) \tau_k \frac{\partial}{\partial \tau_k} - \left(\sum_{C \in K_k} M(C)\right) \tau_i \frac{\partial}{\partial \tau_i}\right] \Phi_{3nj} = 0.$$

Put $G_L = 0$ if $L \notin E_G$. For $L \in E_G$ we obtain the recursion relations:

$$\sum_{L' \in L(K_i)} \epsilon(L')(l_k(L) - l_k(L')) N_{L-L'} G_{L-L'} = \sum_{L'' \in L(K_k)} \epsilon(L'')(l_i(L) - l_i(L'')) N_{L-L''} G_{L-L''}.$$

11. Concluding remarks

Some other known results come out quite easily from the present point of view: symmetries of the coefficients (compared with the symmetries of the graph); simplification of graphs with a branch $j = 0$ or containing a diode symbol linked to a bound branch; formula (25.15) in Jucys and Bandzaitis (1965) expressing a doubly stretched $9j$ coefficient in terms of a $3j$ coefficient.

The explicit formula, equation (16), expresses the CRC as a summation of products without factorization of terms. The usual method for the numerical computation of a CRC, say a $3nj$ coefficient, is to express it as a sum of products of $6j$ coefficients, each $6j$ coefficient being obtained by a summation of products and quotients of factorials. Equation (16) is thus less efficient for numerical calculations.

Appendix. Proof of some relations between sums and products over sets of diagrams

Two relations are of use in § 8. The first one equation (A.3) relates a sum and a product over sets of circuits that pass over a given branch w. It is obtained by calculating a determinant in two different ways. The second relation equation (A.5), which simplifies an infinite product to a finite sum is a consequence of the first one.

Let w be a branch joining vertices u and u' in a Jucys graph G and put $W = \{w\}$. Let μ be a subset of $H'_G \cup \mathscr{R}_W$ and ζ (or correspondingly: ζ') be a set of different paths of the form $[w\, u \ldots u\, w]$ (or: $]w\, u' \ldots u'\, w[$) and such that branch w appears only at the extremities. We designate by $\bar{\zeta}$ and $\bar{\zeta}'$ the sets composed of the reversed paths of ζ and ζ'. We put $v = \mu \cup \zeta \cup \zeta' \cup \bar{\zeta} \cup \bar{\zeta}'$ and we suppose that $\zeta \cup \zeta'$ and $\bar{\zeta} \cup \bar{\zeta}'$ are disjoint. The reversed path of $i \in v - \mu$ is designated by $\bar{i}$. For $i \in \mu$ we define $P(i)$ as a path obtained by opening circuit i at branch w. $P(i)$ will be interpreted as path $]w\, u' \ldots u\, w]$ or $[w\, u \ldots u'\, w[$ which give the same monomial $M(P(i))$. For $i \in v - \mu$ we put $P(i) = i$.

We now define a square matrix x composed of identical columns by putting $x_{ij} = M(P(i))$ for $i \in v, j \in v$.

Matrix x is of rank 1, so:

$$\det(1-x) = 1 - \sum_{i \in v} x_{ii} = 1 - \sum_{i \in \mu} M(P(i)) = 1 + \sum_{i \in \mu} M(i), \tag{A.1}$$

since for $i \in \zeta \cup \zeta'$, $M(\bar{i}) = -M(i)$ (§5, remark (ii)). We also compute: $\ln \det(1-x)$ by equation (7) of § 8.1:

$$\ln \det(1-x) = -\left(\sum_{i_1} N(i_1) + \tfrac{1}{2} \sum_{i_1 i_2} N(i_1, i_2) + \tfrac{1}{3} \sum_{i_1 i_2 i_3} N(i_1, i_2, i_3) + \ldots \right), \tag{A.2}$$

where we put $N(a, b, c \ldots z) = x_{ab} x_{bc} \ldots x_{za}$ and where the sums are over $i_k \in v$. As in § 8 we interpret the monomials $N(a \ldots z)$ in terms of circuits. But here some monomials, like $N(a)$ or $N(a, b, a', c)$ with $a \in \zeta$, $a' \in \zeta$, $b \in \mu$, $c \in \zeta'$ cannot be associated with a circuit. We now show that these monomials cancel. Let $S = (i_1, \ldots, i_m) \in v^m$ and put $i_{m+j} = i_j$ for $j = 1, 2 \ldots m$. If there exist integers k and k' $(1 \leqslant k < k' \leqslant 2m)$ such that both i_k and $i_{k'}$ belong to one of the sets $\zeta \cup \bar{\zeta}$ and $\zeta' \cup \bar{\zeta}'$ and such that $i_j \in \mu$ for $k < j < k'$ we call S non-c. Let s_m be the set of non-c $S \in v^m$ and put $s'_m = v^m - s_m - \mu^m$. Let Δ' (or: Δ) be the set of non-directed circuits that can be separated into circuits from μ (or: and paths from $v - \mu$) (each circuit and path may occur several times). If $S = (i_1 \ldots i_m) \in \mu^m \cup s'_m$

then $N(S) = -M(C)$, where C is the circuit of Δ formed from $P(i_1), P(i_2) \dots P(i_m)$; if $S \in s_m$ there is no interpretation of $N(S)$ in terms of a circuit.

If $S = (i_1, \dots, i_m) \in s_m \cup s'_m = v^m - \mu^m$, k being the first integer such that $i_k \notin \mu$, we define $f(S)$ as being the sequence of $v^m - \mu^m$ obtained from S by changing i_k into $\bar{i}_k$. From the relations $N(S) + N(f(S)) = 0$, $f(s_m) = s_m$ and $f(s'_m) = s'_m$, we get

$$\sum_{S \in s_m} N(S) = \sum_{S \in s'_m} N(S) = 0.$$

If $C \in \Delta'$ and $C' \in \Delta - \Delta'$ are composed of m paths from $P(v)$, then $-M(C)$ appears $m/\pi(C)$ times in $\Sigma_{S \in \mu^m} N(S)$ and $-M(C')$ appears $2m/\pi(C')$ times in $\Sigma_{S \in s'_m} N(S)$; to get rid of this factor 2 we rewrite equation (A.2) as:

$$\ln \det(1-x) = -\sum_{m=1}^{\infty} \frac{1}{m}\left(\sum_{S \in \mu^m} N(S) + \tfrac{1}{2} \sum_{S \in s'_m} N(S)\right) = \sum_{C \in \Delta} \frac{M(C)}{\pi(C)}$$

and by the same method that gives equation (8), we get:

$$\det(1-x) = \prod_{C \in H'_G \cap \Delta} (1 + M(C)).$$

Comparing with equation (A.1) we get:

$$\prod_{C \in H'_G \cap \Delta} (1 + M(C)) = 1 + \sum_{i \in \mu} M(i).$$

In particular for a set V of branches of G by putting $\mu = H'_G \cap \mathscr{R}_W \cap \bar{\mathscr{D}}_V$ and by taking $\zeta \cup \zeta' \cup \bar{\zeta} \cup \bar{\zeta}'$ to be the maximal subset of $\bar{\mathscr{D}}_V$ compatible with the definition of ζ and ζ', we have $\Delta = H_G \cap \mathscr{D}_W \cap \bar{\mathscr{D}}_V$ and:

$$\prod_{C \in H'_G \cap \mathscr{D}_W \cap \bar{\mathscr{D}}_V} (1 + M(C)) = 1 + \sum_{C \in H'_G \cap \mathscr{R}_W \cap \bar{\mathscr{D}}_V} M(C). \tag{A.3}$$

Let A (or correspondingly: A') be the set of diagrams D composed only of different circuits from $H'_G \cap \mathscr{D}_W \cap \bar{\mathscr{D}}_V$ (or: $H'_G \cap \bar{\mathscr{D}}_V$). For a diagram D and a set of diagrams S we define $S \times D$ to be the set of composite diagrams made up of D and of a diagram D' from S. Let us write, for diagrams D and D', $D \sim D'$ if and only if $\mathscr{B}(D) = \mathscr{B}(D')$. It is clear that this is an equivalence relation in A (or: in A') defining a family $\mathscr{F}$ (or: $\mathscr{F}'$) of equivalence classes. If $\mathscr{A} \in \mathscr{F}$ (or $\mathscr{A} \in \mathscr{F}'$), $D \in \mathscr{A}$ and if D is not simple we say that $\mathscr{A}$ is not simple. Let $\mathscr{A}' \in \mathscr{F}'$ be an equivalence class of A' and $D' \in \mathscr{A}'$ such that D' is not simple, branch w appearing more than once in $\mathscr{B}(D')$. $\mathscr{A}'$ can be partitioned into sets of the form $\mathscr{A} \times D''$ with $D'' \in A' \cap \bar{\mathscr{D}}_W$ and $\mathscr{A} \in \mathscr{F}$ (and a set $\mathscr{A} \in \mathscr{F}$ if $D' \in A$) these $\mathscr{A}$ containing diagrams passing several times on branch w. Expanding the product, we rewrite equation (A.3) as:

$$1 + \sum_{\mathscr{A} \in \mathscr{F}} \sum_{D \in \mathscr{A}} M(D) = 1 + \sum_{C \in H'_G \cap \mathscr{R}_W \cap \bar{\mathscr{D}}_V} M(C). \tag{A.4}$$

By comparing diagrams whose set of elements is identical on both sides of equation (A.4) we see that $\Sigma_{D \in \mathscr{A}} M(D) = 0$ if $\mathscr{A} \cap \mathscr{R}_W = \varnothing$ and $\mathscr{A} \in \mathscr{F}$. Since

$$\sum_{D' \in \mathscr{A} \times D''} M(D') = \left(\sum_{D \in \mathscr{A}} M(D)\right) M(D'')$$

we obtain

$$\sum_{D' \in \mathscr{A}'} M(D') = 0$$

if $\mathscr{A}' \in \mathscr{F}'$ is not simple.

From that it follows:

$$\prod_{C \in H'_G \cap \bar{\mathscr{D}}_V} (1 + M(C)) = 1 + \sum_{\mathscr{A}' \in \mathscr{F}'} \sum_{D' \in \mathscr{A}'} M(D') = 1 + \sum_{D \in K_G \cap \bar{\mathscr{D}}_V} M(D). \tag{A.5}$$

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